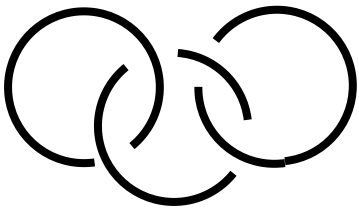
13. LINKS

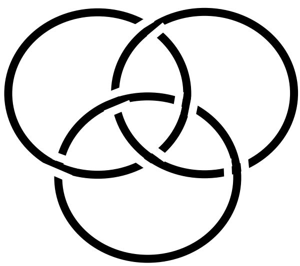
§13.1. Links

A **link** consists of one or more closed loops in $\mathbb{R}^3$ which may be intermingled with one another as well as knotted within themselves. The individual loops are called the **components** of the link. A link with just one component is a knot.

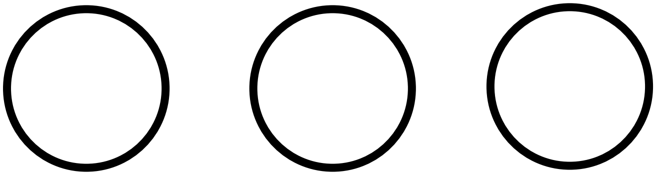
Example 1: Perhaps the most famous link of all is Olympic logo. So that I don't have any copyright problems, here is the corresponding link with just 3 rings. Each ring is one of the components.



Example 2: Another very famous link are the Borromean Rings.



Example 3: A link doesn't need to have any crossings. Three disjoint circles is a very simple link.



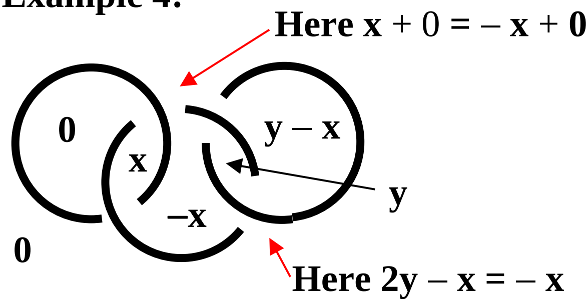
A very crude invariant for a link is the number of components. So we can prove that Example 1 is not equivalent to the Olympic logo because it has 3 components while the Olympic logo has 5.

But in each of the above three examples there are 3 components. Yet, as you can clearly see, they're not equivalent links. You can't manipulate any one to obtain either of the others, but although this is intuitively obvious, you would be hard put to provide a proof.

As with knots, the number of crossings is not an invariant. We can always do a Reidemeister type II move to any link to add a couple of extra crossings.

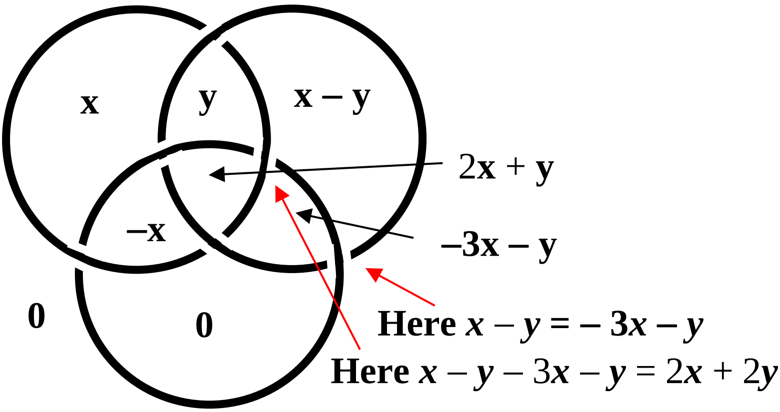
Much of what we have done for knots carries over to links. For example we can define the Alexander Number and the Alexander Group in the same way as for knots. Let's see if the Alexander Group can distinguish these three links.

Example 4:



Alexander Group: $[\mathbf{x}, \mathbf{y} \mid 2\mathbf{x} = 0, 2\mathbf{y} = 0] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
Alexander Number: 4.

Example 5:

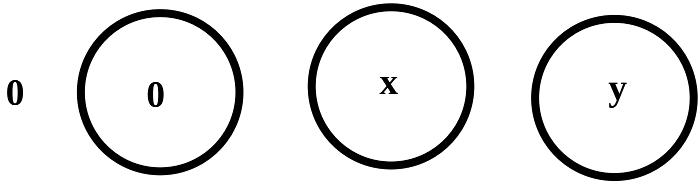


Alexander Group: $[\mathbf{x}, \mathbf{y} \mid 2\mathbf{x} = \mathbf{0}, 4\mathbf{x} + 4\mathbf{y} = \mathbf{0}]$

$$\cong \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix} \cong \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

Alexander Number: 8

Example 6:



Here we have no crossings so we assign two adjacent faces to 0. The outside will be 0 and so will be the inside of one of the rings. The other two rings need separate generators.

Alexander Group: [**x**, **y** |] (no relations) $\cong \mathbb{Z} \oplus \mathbb{Z}$.
Alexander Number: ∞

The Alexander Number proves that all these three links are different.

You can see some important differences between knots and general links. The Alexander Number of a knot is always odd. For a link it can be even, or even infinite.

When it comes to Alexander Modules we run into a real problem. The Alexander Module and Polynomial only apply to oriented links. If there are n components there are 2^n possible orientations. Now if we reversed every orientation we would end up the same Alexander Module and Polynomial. But reversing some but not all of the orientations can result in quite different modules and polynomials.

§13.2. Sum and Direct Sum of Two Links

We can consider two disjoint links L_1 and L_2 as a single link. Here we are using the word ‘disjoint’ to mean that they are completely un-entangled with one another. We could put them in separate drawers in our link cabinet if we wished. Or more mathematically, we could enclose the links in separate disjoint spheres. We call the resulting link $L_1 \oplus L_2$. Note that if L_1 and L_2 are knots, this is different to the knot sum $L_1 + L_2$.

Example 7: Let L_1 be one of the trefoil knots and L_2 be the figure 8 knot. Then $L_1 \oplus L_2$ is the following link with 2 components:

Theorem 1: $A(L_1 \oplus L_2) \cong A(L_1) \oplus A(L_2) \oplus \mathbb{Z}$.

Proof: Proceed with the first link, as usual. The generators and relations will be identical to the case of L_1 alone. When we come to the second link we have 0 for the outside but no further labels. We can only use a pair of 0’s as labels once (remember that this strips off the two unnecessary copies of $\mathbb{Z}$ to reduce the Face Group to the Alexander Group. So where we might normally start with:

0	**0**
$\boldsymbol{x}$	$\boldsymbol{-x}$

if we were considering L_2 alone, we must begin L_2 with:

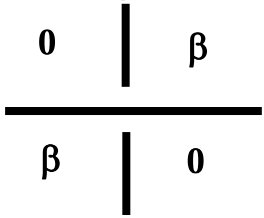
0	**β**
a	**β –** ***a***

Here we are assuming that the labels **a** and **β** have not be used in L_1.

If we put **β** = **0** we would get the equations for $A(L_2)$. So continuing working with the second link of $L_1 \oplus L_2$ the equations would be the same as for $A(L_2)$ alone, with the addition of multiples of **β**.

Colour the faces of L_2 black and white, so that the outside face is white.

Now put **a** = **β** and all subsequent generators that we might need to introduce. The labels for each face will now be 0 for white faces and **β** for black faces.



If we have 3 faces labelled in this way the 4th would have to be labelled in this way. So that at the end the equation will have to reduce to $0\boldsymbol{\beta} = 0$.

So if $A(L_1) = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \mid R_1 = R_2 = \ldots = R_h = \mathbf{0}]$ and

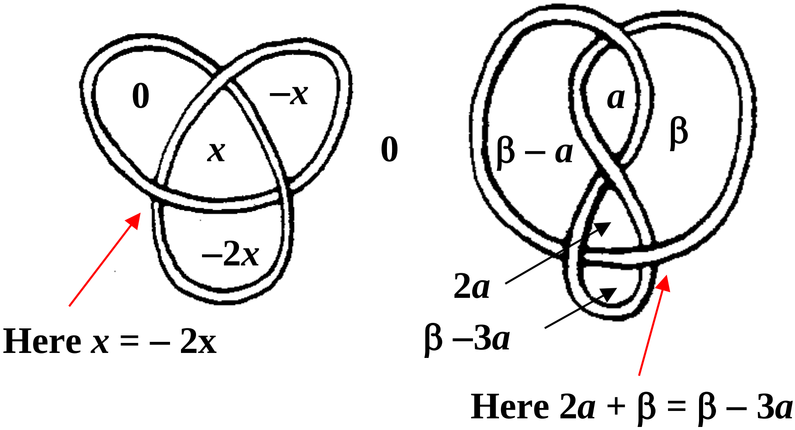
$A(L_2) = [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m \mid S_1 = S_2 = \ldots = S_k = \mathbf{0}]$ then

$$A(L_1 \oplus L_2) = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m, \beta \mid R_1 = R_2 = \ldots = R_h = S_1, S_2, \ldots, S_k = \mathbf{0}].$$

The R_i's will only contain the x_i's and the S_k's will only contain the y_i's. There will be no equation involving β. This will therefore be a generator of infinite order that is independent of the other generators.

Hence $A(L_1 \oplus L_2) \cong A(L_1) \oplus A(L_2) \oplus \mathbb{Z}$.

Example 8: Consider the disjoint union of a trefoil and the figure 8 knot. We will work out the Alexander Group and verify Theorem 1.



So A(trefoil + figure 8)

$$= [x, a, \beta \mid 3x = 5a = 0] \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}.$$

Once we start considering components from one link tangled up with components from another, things get much more complicated.

§13.3. Adding a Ring

Let's begin with the simple case of linking a ring with a knot. Imagine that a knot K is more or less in a horizontal plane, as in the knot map, and the ring starts above K, passes through one of the faces of K, then goes underneath K, bends around and goes up through one of the faces of K before it joins up with the starting point.

Clearly, if those two faces are the same the ring can be pulled free and we have the case of a knot plus a disjoint ring.

Here A(K ⊕ ring) ≅ A(K) ⊕ A(ring) ⊕ ℤ ≅ A(K) ⊕ ℤ.

More interesting is the case where the two faces are different. Let's consider the case where the faces are adjacent in the knot map.

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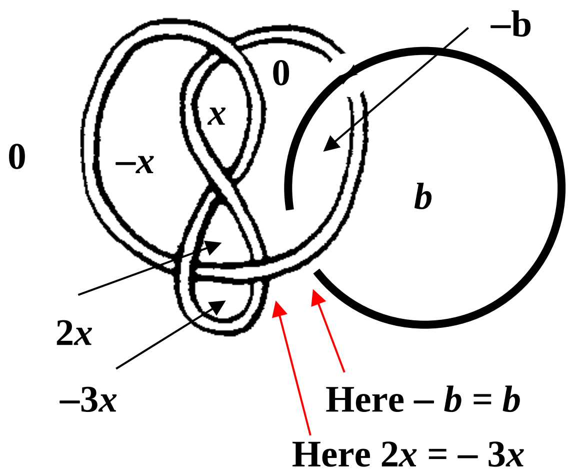
Example 9: Let L be the following link.



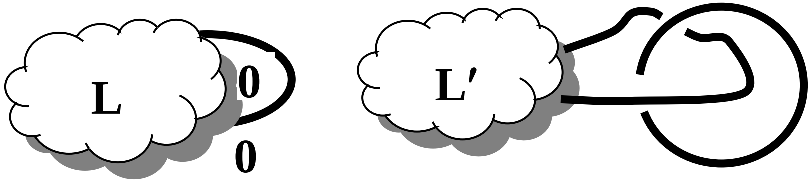
Hence A(L) = [***x***, ***b*** | 5***x*** = 2***b*** = 0] ≅ $\mathbb{Z}_5 \oplus \mathbb{Z}_2$.

The effect has been to add a ℤ term.

Theorem 2: If the link L′ is obtained from the link L by adding a linked ring through adjacent faces in the link map, then

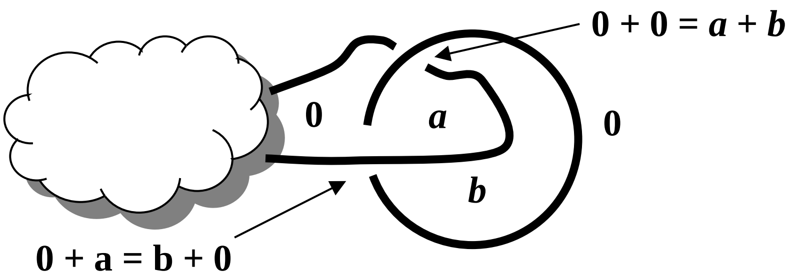
$A(L') \cong A(L) \oplus \mathbb{Z}_2$.

Proof: We can slide the ring around the knot so that one of the faces is the outside. Also we can assign generators to L so that one zero is assigned to the outside and the other to the face that the ring is to be linked to.



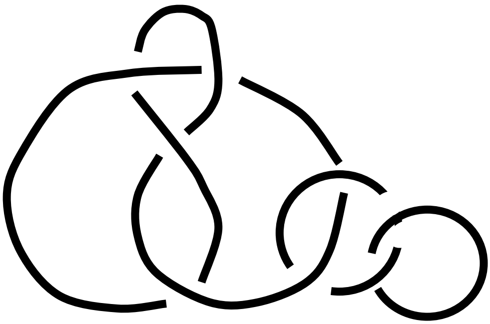
Let A(L) = [… | …]. Here the dots represent the generators and relations.

Then L′ has two additional faces:



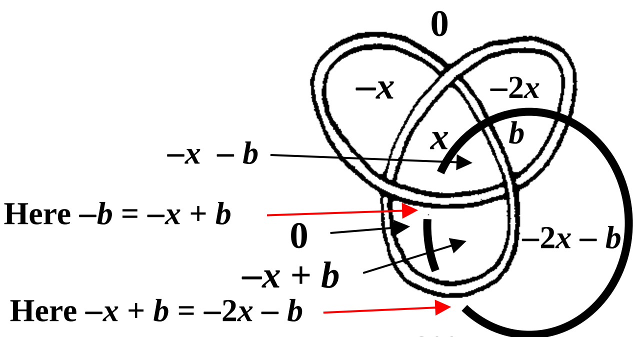
$$\begin{aligned} A(L') &= [\dots, \boldsymbol{a}, \boldsymbol{b} \mid \dots, \boldsymbol{a} + \boldsymbol{b} = \mathbf{0}, \boldsymbol{a} = \boldsymbol{b}] \\ &\cong [\dots, \boldsymbol{a} \mid \dots, 2\boldsymbol{a} = 0)] \\ &\cong [\dots \mid \dots] \oplus [\boldsymbol{a} \mid 2\boldsymbol{a} = 0] \\ &\cong A(L) \oplus \mathbb{Z}_2. \end{aligned}$$

Example 10: This link has 4 components. Its Alexander Group is $\mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.



The basic knot is the figure 8 knot. (Look at it carefully to ensure that it is alternating – otherwise it could be a trefoil in disguise.) The Alexander group is $\mathbb{Z}_5$. Adding the first ring we get $\mathbb{Z}_5 \oplus \mathbb{Z}_2$. Now adding the second ring we get the Alexander Group to be $\mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It wouldn't matter whether we linked this second ring around the original figure 8 knot or the separate ring.

Now let us consider a more complicated case, where the ring passes through faces that is not adjacent.



Hence the Alexander Group is $\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$

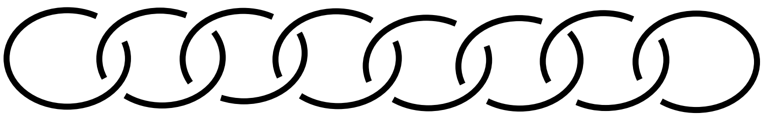
$$\cong \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \cong \mathbb{Z}_4.$$

§13.4. Chains

A **chain** is a link in which the components C_1, C_2, …, C_n are, individually, unknots, and where, for $i = 1, 2, \ldots, n - 1$, C_i is linked to C_{i+1} with one over crossing and one under crossing.

The usual word for each component is 'link' and we talk about the chain being made up of separate 'links'. But we've been calling the whole chain a link. To avoid this confusion we'll use everyday language in this context. We'll refer to the whole object of study as a 'chain', rather than a 'link' and each separate component as a 'link', rather than a 'component'.

Example 11:



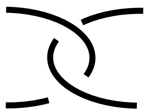
This is a chain with 8 links. By Theorem 2, the Alexander Group for this chain is:

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,

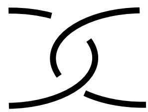
which we write as $\mathbb{Z}_2{}^7$, meaning the direct sum of 7 copies of $\mathbb{Z}_2$.

In general, the Alexander group of a chain with *n* links is $\mathbb{Z}_2{}^{n-1}$, the direct sum of $n - 1$ copies of $\mathbb{Z}_2$.

Now there are two ways that a pair of adjacent links can occur in a projection of a chain and we'll refer to these as positive and negative connections as follows.



+ve connection



–ve connection

It's obvious that this distinction only occurs at the level of projections because both are equivalent for a chain. One link can be rotated relative to the other to achieve the other projection and so the Alexander group of a chain is independent of positive and negative connections. That is, until the ends of the chain are joined. (We call this a **closed chain**.)

The following theorem was obtained by a Macquarie University student in 2004 while on a vacation scholarship.

Theorem 3 (SIMON BYRNE):

Suppose a chain with $n \geq 2$ links has its ends joined. Let m be the absolute difference between the number of positive and negative connections in some projection in which the only crossings are from one link to the next.

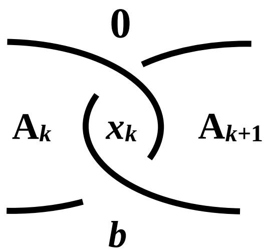
The Alexander group of this closed chain is $\mathbb{Z}_2{}^{n-2} \oplus \mathbb{Z}_{2m}$ if $m > 0$ and $\mathbb{Z}_2{}^{n-2} \oplus \mathbb{Z}$ if $m = 0$.

(Here $\mathbb{Z}_2{}^{n-2}$ denotes the direct sum of $n - 2$ copies of $\mathbb{Z}_2$.)

Proof: Suppose we have a chain with n links. Make a projection in which the chain, ignoring the fact that it is made up of links, does not cross itself. That is, the only crossings are those that arise from links connecting adjacent ones. There are $2n + 2$ faces. One is the region outside the chain, which we shall label as 0. Another is the inside of the chain, which we shall label as b.

Then there are n faces, being the regions surrounded by pieces of adjacent links. We shall label these $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$. Finally there are n faces that form the centre part of the links which we shall label as $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$.

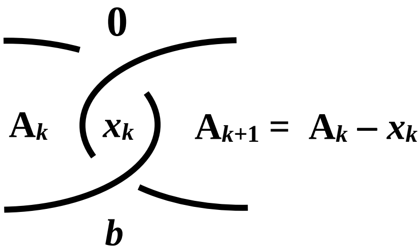
For a positive connection we have:



$$\mathbf{A}_{k+1} = \mathbf{A}_k + \boldsymbol{x}_k$$

Here we have the relation $\mathbf{A}_k + \boldsymbol{b} = \mathbf{A}_k + 2\boldsymbol{x}_k$ which simplifies to $2\boldsymbol{x}_k = \boldsymbol{b}$.

For a negative connection we have:



Here we have the relation $\mathbf{A}_k + \boldsymbol{x}_k = \mathbf{A}_k - \boldsymbol{x}_k + \boldsymbol{b}$ which again simplifies to $2\boldsymbol{x}_k = \boldsymbol{b}$.

So far we have only set the generator for one face equal to $\mathbf{0}$. We now set $\mathbf{A}_1 = \mathbf{0}$.

Then for each k:

$$\mathbf{A}_{k+1} = \sum_{i=1}^{k} c_i \boldsymbol{x}_i$$ where $c_i = 1$ if $\boldsymbol{x}_i$ corresponds to a positive connection and –1 if it corresponds to a negative one.

But, because the n'th link connects to the first, this will give $\sum_{i=1}^{k} c_i \boldsymbol{x}_i = \mathbf{A}_1 = \mathbf{0}$.

The Alexander Group is thus

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n, \boldsymbol{b} \mid \sum_{i=1}^{k} c_i \mathbf{x}_i = \mathbf{0}, 2\boldsymbol{x}_1 = 2\boldsymbol{x}_2 = \ldots 2\boldsymbol{x}_n = \boldsymbol{b}]$$

Multiplying the first relation by 2 and using the remaining ones we deduce the additional relation

$$\left(\sum_{i=1}^{n} c_i\right) \boldsymbol{b} = 0.$$

Hence the Alexander Group is

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n-1}, \boldsymbol{b} \mid m\boldsymbol{b} = 0, 2\boldsymbol{x}_1 = 2\boldsymbol{x}_2 = 2\boldsymbol{x}_{n-1} = \boldsymbol{b}]$$

where $m = \text{ABS}\left(\sum_{i=1}^{n} c_i\right)$.

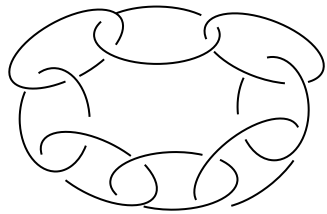
Writing $\boldsymbol{y}_i = \boldsymbol{x}_i - \boldsymbol{x}_1$, for $i = 2, \ldots, n$ we can write this as:

$$[\boldsymbol{x}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{n-1}, \boldsymbol{b} \mid m\boldsymbol{b} = 0, 2\boldsymbol{x}_1 = \boldsymbol{b}, 2\boldsymbol{y}_2 = \ldots = 2\boldsymbol{y}_{n-1} = \mathbf{0}]$$

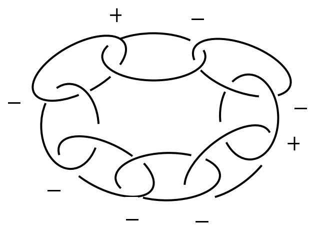
$$\cong [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n-1} \mid 2m\boldsymbol{x}_1 = 2\boldsymbol{y}_2 = 2\boldsymbol{y}_{n-1} = \mathbf{0}]$$

$$\cong \mathbb{Z}_2^{\,n-2} \oplus \mathbb{Z}_{2m} \text{ if } m > 0 \text{ and } \mathbb{Z}_2^{\,n-2} \oplus \mathbb{Z} \text{ if } m = 0.$$

Example 12: Find the Alexander group of the following closed chain.



Solution: There are eight links and so eight connections, of which 2 are positive and 6 are negative. Using the terminology of Theorem 3, $n = 8$ and $m = 6 - 2 = 4$. Hence the Alexander group is $\mathbb{Z}_2^6 \oplus \mathbb{Z}_8$.



EXERCISES FOR CHAPTER 13

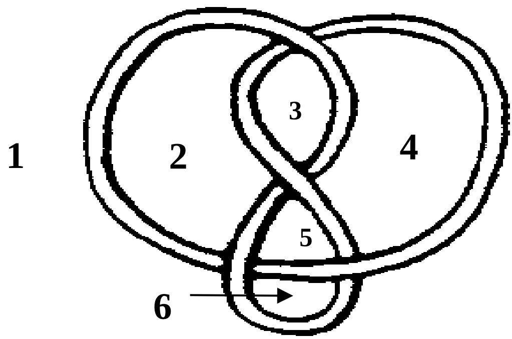
Exercise 1:

Appendix B contains all links with up to 7 crossings, together with their Alexander Groups. Verify that they are as stated.

Exercise 2:

Take a strip of paper and tie a trefoil knot in it, joining up the ends so as to make the surface homeomorphic to a cylinder. Now cut this cut this strip along its length to make a link with two components. Find the Alexander group of this link.

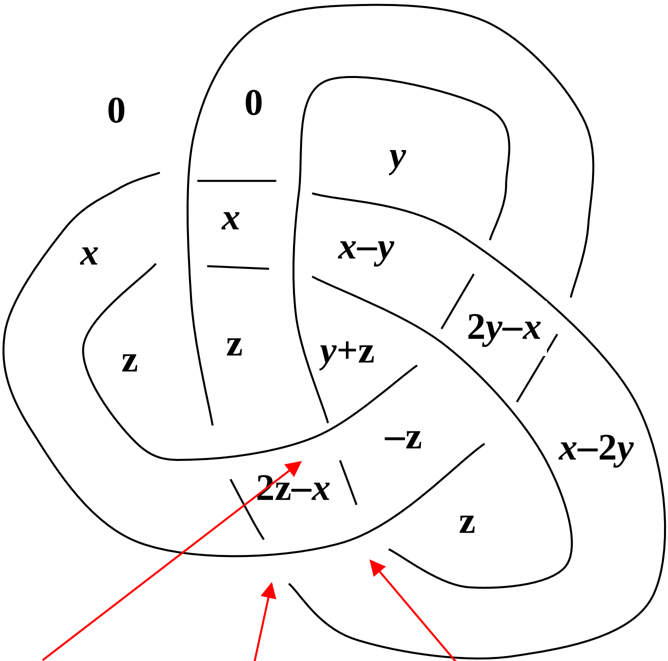
Exercise 3: The following is a knot map of the figure 8 knot, with the 6 faces numbered. Construct a table where the i-j entry gives the Alexander Group of the link obtained from taking a rope down through face i and up through face j and a table that gives the corresponding Alexander Number.



SOLUTIONS FOR CHAPTER 13

Exercise 1: The answers are contained in Appendix B.

Exercise 2: The answer may depend on how many twists are introduced before joining the two ends of the knotted strip.



$y + 2\mathbf{z} = \mathbf{z} - x$

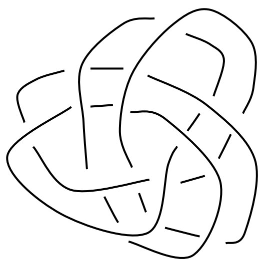
i.e. $x + y + \mathbf{z} = \mathbf{0}$ $\quad 2\mathbf{z} = \mathbf{x} - 2\mathbf{y}$ $\quad \mathbf{z} - x = x - 2y + \mathbf{z}$

i.e. $x - 2y - 2\mathbf{z} = \mathbf{0}$ $\quad$ i.e. $2x - 2y = \mathbf{0}$

The Alexander Group is $\cong \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -2 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & -3 \\ 0 & -4 & -2 \end{bmatrix}$

$$\cong \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 \\ 0 & 6 \end{bmatrix} \cong \mathbb{Z}_6.$$

You may have a couple of twists, as in the following diagram, but the Alexander group would still be $\mathbb{Z}_6$.



There are other, less likely, possibilities which could lead to different Alexander groups.

Exercise 3: Now A(figure 8) $\cong \mathbb{Z}_5$.
Let A(i, j) denote the Alexander Group of the link where the ring passes around faces i, j.
If $i = j$ we have can pull the ring off and we have the situation of a disjoint ring. Hence by Theorem 1, Then A(i, i) $\cong$ A(figure 8) $\oplus$ A(unknot) + $\mathbb{Z}$

$$\cong \mathbb{Z}_5 \oplus 1 \oplus \mathbb{Z} \cong \mathbb{Z}_5 \oplus \mathbb{Z}.$$

If faces i, j are adjacent in the map of the figure 8 knot then, by Theorem 2, A(i, j) $\cong \mathbb{Z}_5 \oplus \mathbb{Z}_2$.

Clearly A(i, j) ≅ A(j, i) for all i, j. This drastically reduces the number of cases we have to compute.

Alexander Groups A(i, j):

	1	**2**	**3**	**4**	**5**	**6**
1	**A**	B	C	B	B	B
2	B	A	B	D	B	D
3	C	B	A	B	C	B
4	B	D	B	A	B	C
5	B	B	C	B	A	B
6	B	D	B	C	B	A

where A = $\mathbb{Z}_5 \oplus \mathbb{Z}$,

B = $\mathbb{Z}_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{10}$,

C = $\mathbb{Z}_4 \oplus \mathbb{Z}_3 = \mathbb{Z}_{12}$,

D = $\mathbb{Z}_{11} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{22}$.

Alexander Numbers:

	1	**2**	**3**	**4**	**5**	**6**
1	∞	10	12	10	10	10
2	10	∞	10	22	10	22
3	12	10	∞	10	12	10
4	10	22	10	∞	10	10
5	10	10	12	10	∞	10
6	10	22	10	12	10	∞